arXiv:math/9810176v2 [math.GN] 7 Nov 1998

An answer to a question of Pyrih*

Julian Dontchev
Department of Mathematics
University of Helsinki
PL 4, Yliopistonkatu 5
00014 Helsinki
Finland

A topological space (X, τ) is called *extremally disconnected* [4] if the closure of every open set is open. Recently, Pyrih [3] defined a space X to be *open-normal* if for any two disjoint open sets A and B there exist disjoint closed sets F_A and F_B such that $A \subset F_A$ and $B \subset F_B$. He asked if every open-normal space is discrete. The answer to his question is no.

Proposition 1. *A topological space (X, τ) is open-normal if and only if it is extremally disconnected.*

Proof. Assume first that X is open-normal. Let A be an open subset of X. Then, A and $B = X \setminus \text{cl}(A)$ are disjoint open sets and hence there exist disjoint closed sets F_A and F_B such that $A \subset F_A$ and $B \subset F_B$. Clearly, $F_A = \text{cl}(A)$ and $F_B = B$. Hence, $\text{cl}(A) = X \setminus F_B$ is open. Thus, X is extremally disconnected.

Assume next that X is extremally disconnected. If A and B are disjoint open subsets of X, then $\text{cl}(A)$ and $X \setminus \text{cl}(A)$ are disjoint open sets containing A and B respectively. □

It is well-known that there exist extremally disconnected non-discrete spaces: the Stone-Čech compactification of every discrete space, etc. A detailed bibliography on the recent progress of the study of extremally disconnected spaces may be found in [1].

In [3], Pyrih proved that every extremally disconnected metric space is discrete. A much better result exists. In [2], Gleason proved that every convergent sequence $(x_1, x_2, \ldots)$ of an

*1991 Math. Subject Classification — Primary: 54G05; Secondary: 54G99, 54H05.
Key words and phrases — extremally disconnected, open-normal, discrete.

extremally disconnected Hausdorff space is stationary. Recall that a sequence $(x_1, x_2, \ldots)$ is called *stationary* if for some n, we have $x_n = x_{n+1} = x_{n+2} = \ldots$. As a consequence of Gleason's result we have the following:

Proposition 2. *Every sequential Hausdorff space which is extremally disconnected is discrete.*

Note that the following implications hold and none of them is reversible:

Metric $\Rightarrow$ First countable and Hausdorff $\Rightarrow$ Fréchet and Hausdorff $\Rightarrow$ Sequential and Hausdorff.

References

[1] J. Dontchev and D. Rose, Extremal disconnectedness modulo dual filtrations, *Math. Pannonica*, **9** (1998), 33-46.

[2] A.M. Gleason, Projective topological spaces, *Illinois J. Math.*, **2** (1958), 482–489.

[3] P. Pyrih, A space where disjoint open sets have disjoint closures, *Questions Answers Gen. Topology*, **16** (2) (1998), 133–134.

[4] M.H. Stone, Algebraic characterizations of special Boolean rings, *Fund. Math.*, **29** (1937), 223–302.

E-mail: `dontchev@cc.helsinki.fi`